13.4 Position, Velocity, Acceleration

If ***t = time*** and position is given by

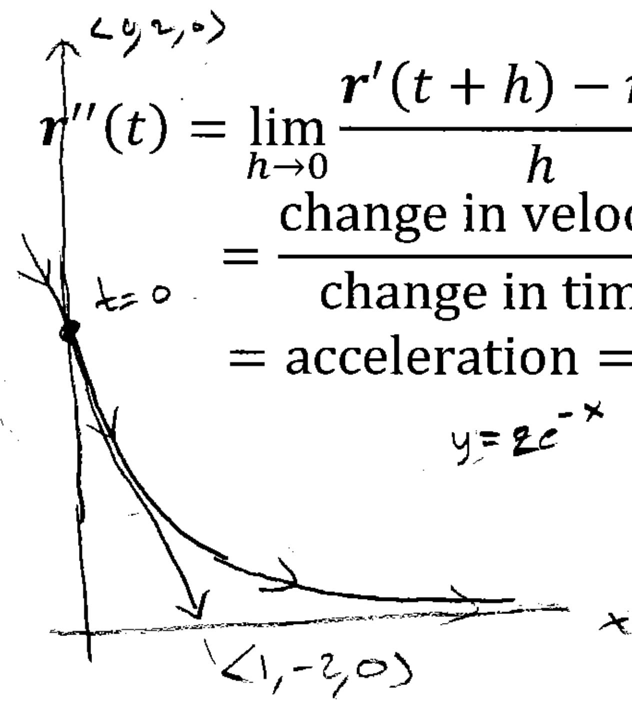
$$\boldsymbol{r}(t) = < x(t), y(t), z(t) >$$

then

$$\boldsymbol{r}'(t) = \lim_{h \to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h}$$
$$= \frac{\text{change in position}}{\text{change in time}}$$
$$= \text{velocity} = \boldsymbol{v}(t)$$

$$|\boldsymbol{r}'(t)| = \frac{\text{change in dist}}{\text{change in time}} = \text{speed}$$

$$\boldsymbol{r}''(t) = \lim_{h \to 0} \frac{\boldsymbol{r}'(t+h) - \boldsymbol{r}'(t)}{h}$$
$$= \frac{\text{change in velocity}}{\text{change in time}}$$
$$= \text{acceleration} = \boldsymbol{a}(t)$$



Example:

Let *t* be **time in seconds** and assume the position of an object (with components in **feet**) is given by

$$\boldsymbol{r}(t) = < t, 2e^{-t}, 0 >$$

Compute

1. $\boldsymbol{r}'(t)$, $|\boldsymbol{r}'(t)|$ and $\boldsymbol{r}''(t)$.
2. $\boldsymbol{r}'(0)$, $|\boldsymbol{r}'(0)|$ and $\boldsymbol{r}''(0)$.

$\vec{r}'(t) = \langle 1, -2e^{-t}, 0 \rangle$

$|\vec{r}'(t)| = \sqrt{1 + 4e^{-2t}}$ ft/sec

$\vec{r}''(t) = \langle 0, 2e^{-t}, 0 \rangle$

$\vec{r}'(0) = \langle 1, -2, 0 \rangle$

$|\vec{r}'(0)| = \sqrt{5}$ ft/sec

$\vec{r}''(0) = \langle 0, 2, 0 \rangle$

HUGE application:
Modeling ANY motion problem.

Newton's 2nd Law of Motion states

Force = mass · acceleration

$$\boldsymbol{F} = m \cdot \boldsymbol{a} \quad \text{, so}$$

$$\boldsymbol{a} = \frac{1}{m} \cdot \boldsymbol{F}$$

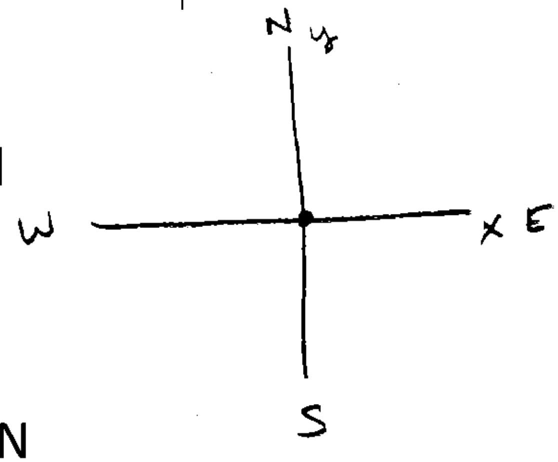
If $\boldsymbol{F} = \langle 0,0,0 \rangle$, then all the forces 'balance out' and the object has no acceleration. (Velocity will remain constant.)

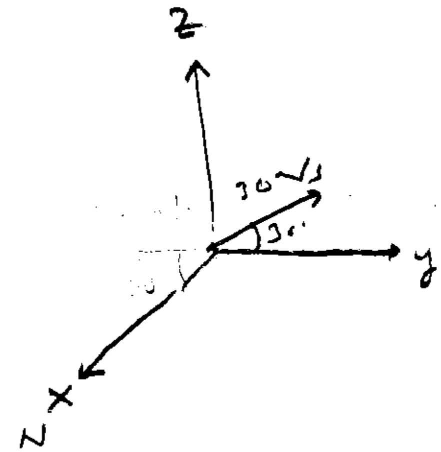
If $\boldsymbol{F} \neq \langle 0,0,0 \rangle$, then acceleration will occur, and we integrate (or solve differential equations) to find velocity and position.

This is how we can model ALL motion problems!

Example:
A ball with mass $m = 0.8$ kg is thrown northward into the air with initial speed of 30 m/sec at an angle of 30 degrees with the ground.

ABOVE VIEW





A west wind applies a steady force of 4 N on the ball (west to east). If you are standing on level ground, where does the ball land?

$$\left.\begin{array}{l}\vec{F}_g = \langle 0, 0, -9.8m \rangle \\ \vec{F}_w = \langle 4, 0, 0 \rangle\end{array}\right\} \vec{F} = \langle 4, 0, -9.8m \rangle$$

$$\vec{a}(t) = \frac{1}{m}\langle 4, 0, -9.8m \rangle$$
$$= \left\langle \frac{4}{0.8}, 0, -9.8 \right\rangle = \langle 5, 0, -9.8 \rangle$$

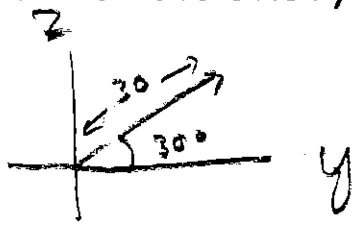
Steps (for *all* motion problems):
1. Forces?
2. Get acceleration.
3. Integrate to get $\vec{v}(t)$ (initial conditions?)
4. Integrate again to get $\vec{r}(t)$ (initial conditions?)

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle 5t + c_1, c_2, -9.8t + c_3 \rangle$$

$$\left\{\begin{array}{l}\vec{v}(0) = \langle 0, 30\cos(30°), 30\sin(30°) \rangle \\ \quad = \langle 0, 30\frac{\sqrt{3}}{2}, 30\frac{1}{2} \rangle = \langle 0, 15\sqrt{3}, 15 \rangle\end{array}\right.$$

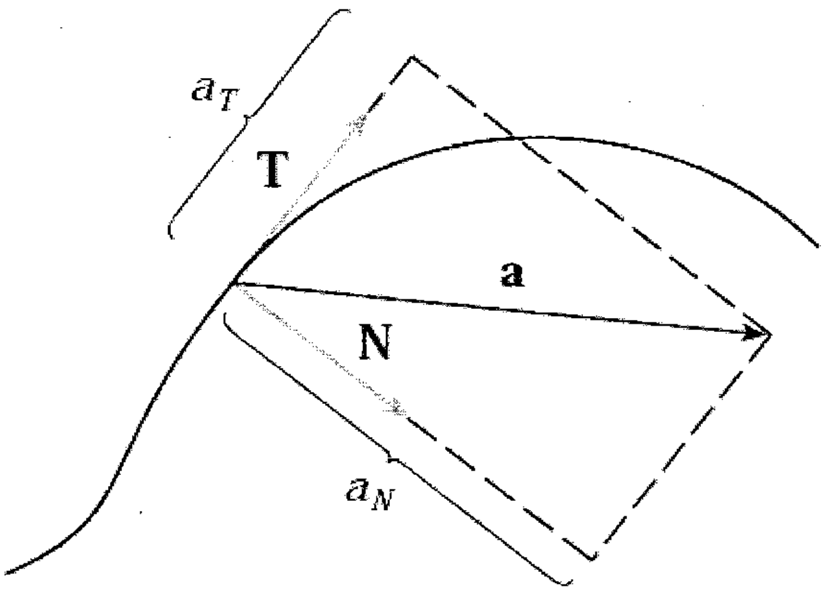
$$\vec{v}(t) = \langle 5t, 15\sqrt{3}, -9.8t + 15 \rangle$$

$$\vec{r}(t) = \langle \tfrac{5}{2}t^2 + d_1, 15\sqrt{3}\,t + d_2, -4.9t^2 + 15t + d_3 \rangle$$
$$\vec{r}(0) = \langle 0, 0, 0 \rangle$$



$$\boxed{\vec{r}(t) = \langle \tfrac{5}{2}t^2, 15\sqrt{3}\,t, -4.9t^2 + 15t \rangle}$$

Measuring and describing acceleration



Recall: $\text{comp}_{\boldsymbol{b}}(\boldsymbol{a}) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}}$ = length.

We define the tangential and normal components of acceleration by:

$a_T = \text{comp}_{\boldsymbol{T}}(\boldsymbol{a}) = \boldsymbol{a}\cdot\boldsymbol{T}$ = tangential

$a_N = \text{comp}_{\boldsymbol{N}}(\boldsymbol{a}) = \boldsymbol{a}\cdot\boldsymbol{N}$ = normal

Note that: $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$

Derivation of interpretation:

Let $\nu(t) = |\vec{\boldsymbol{v}}(t)|$ = speed.

1.$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|} = \frac{\vec{\boldsymbol{v}}(t)}{\nu(t)}$ implies $\vec{\boldsymbol{v}} = \nu\vec{\boldsymbol{T}}$.

2.$\kappa(t) = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{|\vec{\boldsymbol{T}}'|}{\nu(t)}$ implies $|\vec{\boldsymbol{T}}'| = \kappa\nu$.

3.$\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \frac{\vec{\boldsymbol{T}}'}{\kappa\nu}$,implies $\vec{\boldsymbol{T}}' = \kappa\nu\vec{\boldsymbol{N}}$.

Differentiating the first fact above gives

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}' = \nu'\vec{\boldsymbol{T}} + \nu\vec{\boldsymbol{T}}', \text{ so}$$

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}' = \nu'\vec{\boldsymbol{T}} + k\nu^2\vec{\boldsymbol{N}}.$$

Conclusion

$a_T = \nu' = \frac{d}{dt}|r'(t)|$ = "deriv. of speed"

$a_N = k\nu^2$ = curvature · (speed)^2

For computational purposes, we use

$$a_T = \frac{\vec{\boldsymbol{r}}'\cdot\vec{\boldsymbol{r}}''}{|\vec{\boldsymbol{r}}'|} \quad \text{and} \quad a_T = \frac{|\vec{\boldsymbol{r}}'\times\vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|}$$

Example:

$$\vec{\boldsymbol{r}}(t) = < \cos(t)\,, \sin(t)\,, t >$$

Find the tangential and normal components of acceleration.

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

$$\vec{r}\,''(t) = \langle -\cos(t), -\sin(t), 0 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2} \longrightarrow \text{speed} = \underbrace{\sqrt{2}\ \text{ft/sec}}_{\text{CONSTANT}}$$

$$\vec{r}\,'(t) \cdot \vec{r}\,''(t) = \cos(t)\sin(t) - \cos(t)\sin(t) + 0 = 0$$

$$a_T = \frac{\vec{r}\,' \cdot \vec{r}\,''}{|\vec{r}\,'|} = \frac{0}{\sqrt{2}} = 0 \implies \text{SPEED IS NOT CHANGING}$$

$$\vec{r}\,' \times \vec{r}\,'' = \langle 0 - -\sin(t), -(0 - -\cos(t)), \sin^2(t) - -\cos^2(t) \rangle$$

$$= \langle \sin(t), -\cos(t), 1 \rangle$$

$$|\vec{r}\,' \times \vec{r}\,''| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

$$a_N = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|} = \frac{\sqrt{2}}{\sqrt{2}} = 1 = K\,(\text{SPEED})^2 \implies 1 = K(\sqrt{2})^2$$

$$\underbrace{K = 1/2}_{\text{CURVATURE IS CONSTANT}}$$